

Summary on $\bar{m}_{c,b}(\bar{m}_{c,b})$ and precise $f_{D_{(s)},B_{(s)}}$ from heavy-light QCD spectral sum rules *

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Abstract

We summarize recent results obtained in [1] on the running $\bar{m}_{c,b}(\bar{m}_{c,b})$ in the \overline{MS} scheme and $f_{D_{(s)},B_{(s)}}$ using QCD spectral sum rules (QSSR) known to N2LO PT series, including all dimension-six NP condensate contributions in the full QCD theory, an estimate of the N3LO terms based on the geometric growth of the PT series and using the most recent values of the QCD input parameters given in Table 1. The study of the effects of the subtraction scale μ on “different QSSR data” and the use (for the first time) of the Renormalization Group Invariant (RGI) scale independent quark masses in the analysis are emphasized. The estimates [rigorous model-independent upper bounds] reported in Table 2: $f_D/f_\pi = 1.56(5)[\leq 1.68(1)]$, $f_B/f_\pi = 1.58(5)[\leq 1.80(3)]$ and $f_D/f_K = 1.58(4)[\leq 1.63(1)]$, $f_B/f_K = 1.50(3)[\leq 1.61(3.5)]$, improve previous QSSR estimates. The remarkable agreements with some of the experimental data on f_{D,D_s} and with lattice simulations within dynamical quarks confirm both the success of the QSSR semi-approximate approach based on the OPE in terms of the quark and gluon condensates and of the Minimal Duality Ansatz (MDA) for parametrizing the hadronic spectral function which we have tested from the complete data of the J/ψ and Υ systems. The running quark masses $\bar{m}_c(m_c) = 1286(66)$ MeV and $\bar{m}_b(m_b) = 4236(69)$ MeV from $M_{D,B}$ are in good agreement though less accurate than the ones from recent J/ψ and Υ sum rules.

Keywords: QCD spectral sum rules, meson decay constants, heavy quark masses.

1. Introduction and a short historical review

The (pseudo)scalar meson decay constants f_P are of prime interests for understanding the realizations of chiral symmetry in QCD. In addition to the well-known values of f_π and f_K which control the light flavour chiral symmetries, it is also desirable to extract the ones of the heavy-light charm and bottom quark systems with high-accuracy. These decay constants are normalized through the matrix element:

$$\langle 0 | J_{\bar{q}Q}^P(x) | P \rangle = f_P M_P^2 : J_{\bar{q}Q}^P(x) \equiv (m_q + M_Q) \bar{q}(i\gamma_5)Q, \quad (1)$$

where: $J_{\bar{q}Q}^P(x)$ is the local heavy-light pseudoscalar current; $q \equiv d, s$; $Q \equiv c, b$; $P \equiv D_{(s)}, B_{(s)}$ and where f_P is related to the leptonic width $\Gamma(P^+ \rightarrow l^+ \nu_l)$. Since the original works of NSVZ [7] based on QCD spectral sum rules (QSSR) [2]¹, a large amount of QSSR works have been devoted to give bounds on f_P [8–10] and to estimate their values [11–13]. More recent works including N2LO PT corrections have been derived later on in full QCD [14–16] and in HQET [17]. Different earlier papers [4, 5] have been scrutinized in [5, 11], where Narison found that the apparent discrepancies between the different QSSR results can be solved if one applies carefully the stability criteria (also called sum rule window) of the results versus the external QSSR variables and continuum threshold t_c . In

this way, and for given values of $m_{c,b}$, he obtained the values: $f_D \approx (1.31 \pm 0.12)f_\pi$, $f_B \approx (1.6 \pm 0.1)f_\pi$, which are independent of the forms of the sum rules used. However, the result has been quite surprising as it indicates a large violation of the heavy quark symmetry scaling predictions [18], where $1/M_Q$ corrections have been estimated numerically in [12]. This “unexpected result” has been confirmed few years later by lattice calculations [19]. In this talk, we present improved estimates and bounds on f_P and on $\bar{m}_{c,b}(m_{c,b})$ obtained recently in [1] in full QCD theory, where the most recent values of the (non-)perturbative QCD parameters given in Table 1 have been used. The expressions of NLO PT in [9, 20], of N2LO PT in [21], of the non-perturbative in [7, 10] and the light quark mass corrections in [9, 10, 22] have been used. The N3LO PT contributions have been estimated by assuming the geometric growth of the series [23] which is dual to the effect of a $1/q^2$ term [24, 25]. Like in [13], the previous PT results obtained in the on-shell scheme is translated to the one in the \overline{MS} -scheme by using the relation between the on-shell and \overline{MS} mass known at present to N3LO [4, 5]. The Renormalization Group Invariant (RGI) s, c, b quark masses introduced by [26] which are scale and (massless) scheme independent have been also used for the first time, while a careful study of the effect of the subtraction scale on each “QSSR data point” has been performed.

2. QCD spectral sum rules (QSSR)

Forms of the sum rules

We shall be concerned with the two-point correlator :

$$\psi_{\bar{q}Q}^P(q^2) = i \int d^4x e^{iq \cdot x} \langle 0 | T J_{\bar{q}Q}^P(x) J_{\bar{q}Q}^P(0)^\dagger | 0 \rangle, \quad (2)$$

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¹For reviews, see e.g. [3–6].

where $J_{\bar{q}Q}(x)$ is the local current defined in Eq. (1). The associated Laplace sum rules (LSR) $\mathcal{L}_{\bar{q}Q}(\tau)$ and its ratio $\mathcal{R}_{\bar{q}Q}(\tau)$ read [2]²:

$$\begin{aligned}\mathcal{L}_{\bar{q}Q}(\tau, \mu) &= \int_{(m_q+M_Q)^2}^{t_c} dt e^{-t\tau} \frac{1}{\pi} \text{Im}\psi_{\bar{q}Q}^P(t, \mu), \\ \mathcal{R}_{\bar{q}Q}(\tau, \mu) &= \frac{\int_{(m_q+M_Q)^2}^{t_c} dt t e^{-t\tau} \frac{1}{\pi} \text{Im}\psi_{\bar{q}Q}^P(t, \mu)}{\int_{(m_q+M_Q)^2}^{t_c} dt e^{-t\tau} \frac{1}{\pi} \text{Im}\psi_{\bar{q}Q}^P(t, \mu)},\end{aligned}\quad (3)$$

where μ is the subtraction point which appears in the approximate QCD series when radiative corrections are included. The ratio of sum rules $\mathcal{R}_{\bar{q}Q}(\tau, \mu)$ is useful, as it is equal to the resonance mass squared, in the Minimal Duality Ansatz (MDA) parametrization of the spectral function:

$$\frac{1}{\pi} \text{Im}\psi_{\bar{q}Q}^P(t) \simeq f_P^2 M_P^4 \delta(t - M_P^2) + \text{“QCD cont.”} \theta(t - t_c), \quad (4)$$

where f_P is the decay constant defined in Eq. (1) and the higher states contributions are smeared by the “QCD continuum” coming from the discontinuity of the QCD diagrams and starting from a constant threshold t_c . We shall also use for the B -meson, the moments obtained after deriving $n + 1$ -times the two-point function and evaluated at $Q^2 = 0$ (MSR) [2], where an expansion in terms of the on-shell mass M_b can be used. They read:

$$\begin{aligned}\mathcal{M}_{\bar{q}b}^{(n)}(\mu) &= \int_{(m_q+M_b)^2}^{t_c} \frac{dt}{t^{n+2}} \frac{1}{\pi} \text{Im}\psi_{\bar{q}b}^P(t, \mu), \\ \mathcal{R}_{\bar{q}b}^{(n)}(\mu) &= \frac{\int_{(m_q+M_b)^2}^{t_c} \frac{dt}{t^{n+2}} \frac{1}{\pi} \text{Im}\psi_{\bar{q}b}^P(t, \mu)}{\int_{(m_q+M_b)^2}^{t_c} \frac{dt}{t^{n+3}} \frac{1}{\pi} \text{Im}\psi_{\bar{q}b}^P(t, \mu)}.\end{aligned}\quad (5)$$

Test of the Minimal Duality Ansatz (MDA) from J/ψ and Υ

The MDA presented in Eq. (4), when applied to the ρ -meson reproduces within 15% accuracy the ratio $\mathcal{R}_{\bar{d}d}$ measured from the total cross-section $e^+e^- \rightarrow I = 1$ hadrons data (Fig. 5.6 of [5]), while in the case of charmonium, M_ψ^2 from $\mathcal{R}_{\bar{c}c}^{(n)}$ has been compared with the one from complete data where a remarkable agreement for higher $n \geq 4$ values (Fig. 9.1 of [5]) has been found. Tests of MDA from the J/ψ and Υ systems have been also done in [1]. Taking $\sqrt{t_c} \simeq M_{\Upsilon(2S)} - 0.15$ GeV, we show (for instance) the ratio between $\mathcal{L}_{\bar{b}b}^{\text{exp}}$ and $\mathcal{L}_{\bar{b}b}^{\text{dual}}$ in Fig. 1(a) for LSR and $\mathcal{M}_{\bar{b}b}^{(n)\text{exp}}$ and $\mathcal{M}_{\bar{b}b}^{(n)\text{dual}}$ for MSR in Fig. 1(b) for the Υ systems indicating that for heavy quark systems the rôle of the QCD continuum will be smaller than in the case of light quarks and the exponential weight or high number of derivatives suppresses efficiently the QCD continuum contribution but enhances the one of the lowest ground state in the spectral integral. We have used the simplest QCD continuum expression for massless quarks to order α_s^3 from the threshold t_c [28]³: QCD cont. = $1 + as + 1.5as^2 - 12.07as^3$. One can see

that the MDA, with a value of $\sqrt{t_c}$ around the value of the 1st radial excitation mass, describes quite well the complete data in the region of τ and n where the corresponding sum rules present τ (in units of GeV^{-2}) or n stability [28]:

$$\tau^\psi \simeq (1.3 \sim 1.4), \quad \tau^\Upsilon \simeq (0.2 \sim 0.4), \quad n^\Upsilon \simeq (5 \sim 7), \quad (6)$$

as we shall see later on. Moreover, MDA has been also used in [29] (called Minimal Hadronic Ansatz in this paper) in the context of large N_c QCD, where it provides a very good approximation to the observables one compute.

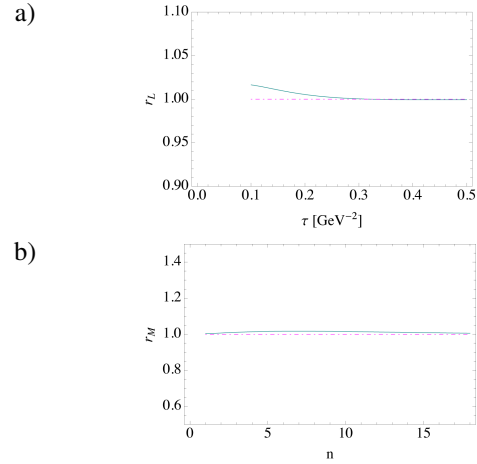


Figure 1: **a)** τ -behaviour of the ratio of $\mathcal{L}_{\bar{b}b}^{\text{exp}} / \mathcal{L}_{\bar{b}b}^{\text{dual}}$ for $\sqrt{t_c} = M_{\Upsilon(2S)} - 0.15$ GeV. The red dashed curve corresponds to the strict equality for all values of τ ; **b)** the same as a) but for $\mathcal{M}_{\bar{b}b}^{(n)\text{exp}} / \mathcal{M}_{\bar{b}b}^{(n)\text{dual}}$ versus the number of derivatives n .

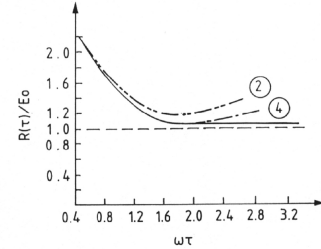


Figure 2: **a)** τ -behaviour of $\mathcal{R}(\tau)$ normalized to the ground state energy E_0 for the harmonic oscillator. 2 and 4 indicate the number of terms in the approximate series.

Optimal results from stability criteria

Using the theoretical expressions of $\mathcal{L}_{\bar{d}Q}^{\text{th}}$ or $\mathcal{M}_{\bar{d}b}^{(n)\text{th}}$, and parametrizing its experimental side $\mathcal{L}_{\bar{d}Q}^{\text{exp}}$ or $\mathcal{M}_{\bar{d}b}^{(n)\text{exp}}$ by the MDA in Eq. (4), one can extract the decay constant f_P and the RGI quark mass \hat{m}_Q . In principle the equality $\mathcal{L}_{\bar{d}Q}^{\text{th}} = \mathcal{L}_{\bar{d}Q}^{\text{exp}}$ should be satisfied for any values of the external (unphysical) set of variables (τ, t_c) , if one knows exactly $\mathcal{L}_{\bar{d}Q}^{\text{th}}$ and $\mathcal{L}_{\bar{d}Q}^{\text{exp}}$. Here, we make an analogy with the harmonic oscillator, where the ratio of moments $\mathcal{R}_{\bar{d}Q}$ is a function of the imaginary time variable τ and where one knows the exact and approximate results. One can find [30] that the exact energy E_0 of the ground state can be approached from above by the approximate series (see Fig. 2). At the minimum or inflexion point (stability) of the curves, one has a ground state dominance. For small time (large Q^2),

²Radiative corrections to the exponential sum rules have been first derived in [27], where it has been noticed that the PT series has the property of an Inverse Laplace transform.

³We have checked that the spectral function including complete mass corrections give the same results.

all level contributes, while for large time (small Q^2) the series breakdown. We shall apply this stability criterion inspired from quantum mechanics in our analysis. In principle, the continuum threshold $\sqrt{t_c}$ in Eq. (4) is a free parameter, though one expects its value to be around the mass of the 1st radial excitation because the QCD spectral function is supposed to smear all the higher state contributions in the spectral integral as explicitly shown previously in Section 2. In order to avoid the model-dependence on the results, Refs. [4, 5, 11–14] have considered the conservative range of t_c -values where one starts to have τ - or n -stability until which one reaches a t_c -stability where the contribution of the lowest ground state to the spectral integral completely dominates. For the D and B mesons, this range is:

$$t_c^D \simeq (5.5 \sim 9.5) \text{ GeV}^2, \quad t_c^B \simeq (33 \sim 45) \text{ GeV}^2. \quad (7)$$

3. The QCD input parameters

We shall use the QCD parameters (with obvious notations) and their values given in Table 1, where $\rho \simeq 2$ indicates the deviation from the four-quark vacuum saturation. From the running $\bar{m}_{q,Q}$ quark parameters, one can deduce the corresponding RGI quantities $\hat{m}_{q,Q}$ and $\hat{\mu}_q$ [26] known to order α_s^3 [4, 5, 31]: $\bar{m}_{q,Q}(\tau) = \hat{m}_{q,Q}(-\beta_1 a_s)^{-2/\beta_1} \times C(a_s)$, $\langle \bar{q}q \rangle(\tau) = -\hat{\mu}_q^3(-\beta_1 a_s)^{2/\beta_1}/C(a_s)$, $\langle \bar{q}g\sigma Gq \rangle(\tau) = -M_0^2 \hat{\mu}_q^3(-\beta_1 a_s)^{1/3\beta_1}/C(a_s)$, where $\beta_1 = -(1/2)(11 - 2n_f/3)$ is the first coefficient of the β function for n_f flavours; $a_s \equiv \alpha_s(\tau)/\pi$. The QCD correction factor $C(a_s)$ is $1 + 1.1755a_s + 1.5008a_s^2 + \dots$, for $n_f = 5$ flavours and shows a good convergence. Therefore, the RGI quantities to order α_s^2 (heavy quarks) and to order α_s (light quarks), in units of MeV are: $\hat{m}_c = 1467(14)$, $\hat{m}_b = 7292(14)$, $\hat{m}_s = 128(7)$, $\hat{\mu}_q = 251(6)$.

Table 1: QCD input parameters.

Parameters	Values	Ref.
$\alpha_s(M_\tau)$	0.325(8)	[32–34]
$\bar{m}_s(2)$	96.1(4.8) MeV	average [35]
$\bar{m}_c(m_c)$	1261(12) MeV	average [28]
$\bar{m}_b(m_b)$	4177(11) MeV	average [28]
$\frac{1}{2}\langle \bar{u}u + \bar{d}d \rangle^{1/3}(2)$	$-(275.7 \pm 6.6)$ MeV	[4, 35]
$\langle \bar{s}s \rangle / \langle \bar{d}d \rangle$	0.74(3)	[4, 35, 36]
M_0^2	$(0.8 \pm 0.2) \text{ GeV}^2$	[37–39]
$\langle \alpha_s G^2 \rangle$	$(7 \pm 1) \times 10^{-2} \text{ GeV}^4$	[28, 30, 32, 40, 41]
$\langle g^3 G^3 \rangle$	$(8.2 \pm 1.0) \text{ GeV}^2 \times \langle \alpha_s G^2 \rangle$	[28]
$\rho \langle \bar{q}q \rangle^2$	$(4.5 \pm 0.3) \times 10^{-4} \text{ GeV}^6$	[32, 37, 40]

4. QSSR analysis and results

The QCD expressions of the sum rules are given in [1] to order α_s^2 . We illustrate the analysis for the LSR in Fig. 3 for the D -meson and for the MSR in Fig.4 for the B -meson. The optimal values of $f_{D(s)}$ and $f_{B(s)}$ are obtained in the range of τ and t_c values given in Eqs. (6) and (7). We study the dependence of the results on the values of the subtraction scale μ in Fig. (5).

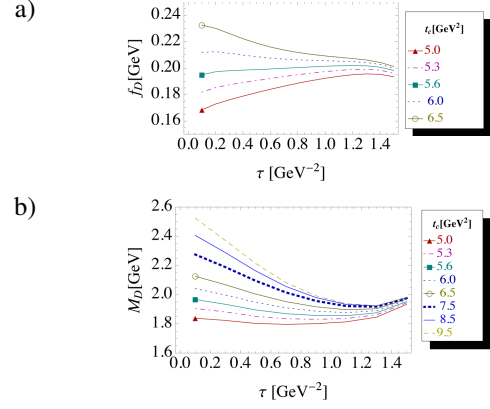


Figure 3: **a)** τ -behaviour of f_D from \mathcal{L}_{dc} for different values of t_c , for a given value of the subtraction point $\mu = \tau^{-1/2}$ and for $\hat{m}_c = 1467$ MeV; **b)** the same as a) but for M_D from \mathcal{R}_{dc} .

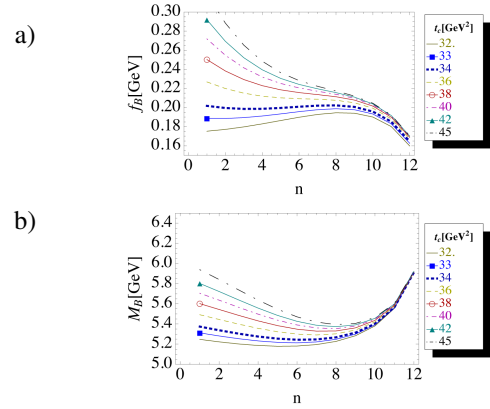


Figure 4: **a)** n behaviour of f_B from MSR for different values of t_c , for $\hat{m}_b = 7292$ MeV and at the subtraction point $\mu = 4$ GeV; **b)** the same as a) but for M_B .

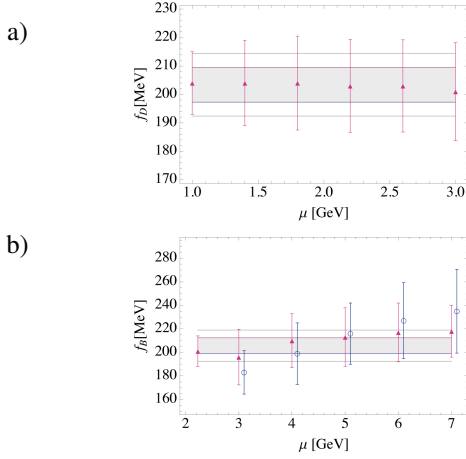


Figure 5: **a)** Values of f_D from LSR at different values of the subtraction point μ and for $\hat{m}_c=1467$ MeV; **b)** Values of f_B from LSR (red triangle) and from MSR (blue open circle) at different values of the subtraction point μ and for $\hat{m}_b=7292$ MeV.

5. Summary and conclusions

Table 2: Results from the open charm and beauty systems in units of MeV and comparison with experimental data and lattice simulations using $n_f = 2$ [43, 44] and $n_f = 3$ [45, 46] dynamical quarks. f_P are normalized as $f_\pi = 130.4$ MeV and $f_K = 156.1(9)$ MeV [47].

Charm	Bottom	Ref.
$\bar{m}_c(\bar{m}_c)$	$\bar{m}_b(\bar{m}_b)$	
1286(66)	4236(69)	This work
1280(40)	4290(140)	ETMC[43]
f_D	f_B	
$204(6) \equiv 1.56(5)f_\pi$	$206(7) \equiv 1.58(5)f_\pi$	This work
$\leq 218.4(1.4) \equiv 1.68(1)f_\pi \leq 235.3(3.8) \equiv 1.80(3)f_\pi$		This work
207(9)	–	Data [47, 48]
212(8)	195(12)	ETMC[43]
–	193(10)	ALPHA [44]
207(4)	190(13)	HPQCD [45]
219(11)	197(9)	FNAL [46]
f_{D_s}	f_{B_s}	
$246(6) \equiv 1.59(5)f_K$	$234(5) \equiv 1.51(4)f_\pi$	This work
$\leq 253.7(1.5) \equiv 1.61(1)f_K \leq 251.3(5.5) \equiv 1.61(4)f_K$		This work
260(5.4)	–	Data [47, 48]
248(6)	232(12)	ETMC[43]
–	219(12)	ALPHA [44]
248(2.5)	225(4)	HPQCD [45]
260(11)	242(10)	FNAL [46]

We have re-extracted the decay constants f_{D,D_s} and f_{B,B_s} and the running quark masses $\bar{m}_{c,b}(\bar{m}_{c,b})$ using QCD spectral sum rules (QSSR), the recent values of the QCD (non-)perturbative parameters given in Table 1 and (for the first time) the scale independent Renormalization Group Invariant (RGI) heavy quark masses in the analysis after translating the on-shell PT expression of the spectral function to the \overline{MS} scheme. We have noticed that f_P are very sensitive to m_Q and decreases when m_Q increases. We have taken the conservative range of t_c in Eq. (7) covering the beginning of the τ - or n -stability until the t_c -stability [Figs. (3) and (4)]. We have carefully studied the effects of the subtraction scale μ on the “QSSR data” [Fig. (5)]. Our final results in Table 2 agree and improve existing QSSR results in the literature. Large mass corrections responsible of

$f_D \simeq f_B$ have been estimated in [12]. f_D and f_{D_s} agree within the errors with the data compiled in [47, 48], while the upper bound on f_{D_s} can already exclude some existing data and theoretical estimates (see e.g. [49] for some attempts to extract $|V_{cd}|$ and $|V_{cs}|$). Our results are compatible (in values and precisions) with lattice simulations including dynamical quarks [43–46], which are not surprising as both methods study the same two-point correlator though evaluated in two different space-times and use the 1st principles of QCD (here is the OPE in terms of the quarks and gluon condensates which semi-approximate non-perturbative confinement). These agreements also confirm the accuracy of the MDA for describing the spectral function in the absence of complete data, which has been tested earlier [4, 5] and in this paper from the charmonium and bottomium systems and in [29] in the large N_c limit of QCD.

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